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BELYI LATTÈS MAPS

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Abstract. In this work, using elementary tools we determine those Lattès maps which are at the same time Belyi maps by explicitly determining their ramification data. It turns out that in the generic case, i.e. when the automorphism group is $\mathbf{Z}/2\mathbf{Z}$, the corresponding family of Lattès maps are Belyi maps if the isogeny is multiplication by two or four. Elliptic curves with extra automorphisms also determine families of Belyi maps. We provide examples of some Belyi Lattès maps together with a formula for such maps which may be used to write Belyi maps of arbitrarily high degree. We conclude the paper with a discussion of the field of definition of such Belyi pairs.

1. Introduction

An algebraic curve, X, admitting a model whose defining equations have algebraic coefficients is referred to as an *arithmetic curve*. A celebrated theorem of Belyi [**1**] states that arithmetic curves admit a *Belyi map*, ϕ; that is, a meromorphic function ramified at most over three points, which are often fixed as 0, 1 and ∞. The pair (X, ϕ) is called a *Belyi pair*. Conversely, if an algebraic curve X admits such a Belyi map, then X is an arithmetic curve [**19**].

The absolute Galois group, $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, acts on the set of Belyi pairs (X, ϕ) by the action of any $\sigma \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ on the coefficients of the defining equation(s) of X and on ϕ. Even though studying a group via its actions on well-known objects is a sound principle, understanding the mentioned action in full generality is widely accepted as a task beyond reach. One is therefore led to work on certain *natural substructures* which are rich enough, in that one may capture $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. It is known that $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ acts on the set of arithmetic elliptic curves faithfully; see [**9**] for a detailed account. More generally, this action is faithful for arbitrary curves of genus g (see [**5**]), a result which was first proven for the genus-zero case [**14**]. There are other families which admit (conjecturally) a faithful action; see for instance [**6, 17, 18**].

A map $\psi\colon \mathbb{P}^1 \to \mathbb{P}^1$ is said to be a quotient of an affine map if it is semi-conjugate to an affine self-map of $\mathbf{C}/\Omega$, where Ω is an additive subgroup of $\mathbf{C}$; see Section 2 for precise definitions. When Ω is of rank one, all such maps are Belyi maps. This motivates the analogous question for the rank-two case. To this end, we determine Lattès maps which are at the same time Belyi maps using elementary tools. This approach leads also to an explicit description of associated Belyi pairs. We find that whenever $\mathbf{C}/\Omega$ does not have

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extra automorphisms, the only isogeny that induces a Belyi map is multiplication by two (and its translates). For each elliptic curve with an extra automorphism, we obtain families of Belyi maps parametrized by the endomorphisms of the corresponding elliptic curve. Using our approach, one may compute Belyi maps of arbitrary high degree, which we exemplify. We then address the question of determining the field of definition of Belyi Lattès maps and the action of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ on Belyi Lattès maps. Let us remark that results in the same vein are obtained by Guralnick, Müller and Saxl [**7**], where the authors study a classification problem referred to as exceptionality of rational maps. Indeed, several Belyi-type maps stemming from different underlying structures are obtained by the authors. Their results and ours agree in the particular case of Lattès maps.

2. Fundamental notions

This section is devoted to defining and collecting some properties of Lattès maps which are particular cases of quotients of affine maps. Our main aim is to state a classification result (Theorem 2.1).

2.1. Quotients of affine maps

A map $\psi\colon \mathbf{C} \longrightarrow \mathbf{C}$ of the form $\psi(z) = \lambda z + \mu$, where $\lambda, \mu \in \mathbf{C}$ with $\lambda \neq 0, 1$, is called an *affine map* of $\mathbf{C}$. We define a map $\phi\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ to be the *quotient of an affine map* if there is a commutative diagram of holomorphic maps:

$$\begin{array}{ccc} \mathbf{C}/\Omega & \xrightarrow{\ \psi\ } & \mathbf{C}/\Omega \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^1 & \xrightarrow{\ \phi\ } & \mathbb{P}^1 \end{array} \tag{1}$$

where Ω is a non-trivial discrete subgroup of $\mathbf{C}$ and π is a non-constant finite map. As ψ is required to be Ω-periodic, we have $\lambda\Omega \subseteq \Omega$. This inclusion ensures the holomorphicity of the map ϕ; see [**3**, Theorem 3.20] for a proof. From now on, we consider lattices up to homotheties. If the rank of Ω is one then $\Omega \cong \mathbf{Z}$, and if the rank of Ω is two then Ω is generated as a $\mathbf{Z}$-module by 1 and τ for some τ in the fundamental region of the action of $\mathrm{PSL}_2(\mathbf{Z})$ on the upper half-plane, $\mathbb{H}$. This means, in the rank-one case $\lambda \in \mathbf{Z}$ and in the rank-two case λ is an eigenvalue of an integral non-degenerate 2×2 matrix (whose eigenvector is $(1\ \tau)^t$, where $\Omega = \langle 1, \tau \rangle$) and hence is a quadratic algebraic integer. In particular, for $\Omega = \mathbf{Z}[\sqrt{-1}]$ or $\Omega = \mathbf{Z}[\zeta_3]$, $\lambda \in \Omega$, where $\zeta_k = \exp(2\pi\sqrt{-1}/k)$.

For simplicity of the exposition, we will assume $\mu = 0$. If Ω is generated by one element, then, by a change of variable, we may assume that the generator is 2π. The commutativity of the above diagram implies that $\phi(\pi(z)) = \pi(\lambda z)$. For $\pi(z) = e^{iz}$ we necessarily have $\phi(w) = w^\lambda$. If $\pi(z) = e^{iz} + e^{-iz}$ then $\phi(w + w^{-1}) = w^\lambda + w^{-\lambda}$. Functions satisfying this functional equation are called Chebyshev polynomials. Up to conjugation, these are the only possibilities in the rank-one case [**13**, Lemma 3.8]. Notice that both w^λ and Chebyshev polynomials are Belyi maps, i.e. ramified at most over three points; see [**10**, Section 1.4.2].

2.2. *Lattès maps*

From now on, we assume that Ω is of rank two generated by 1 and τ so that $\mathbf{C}/\Omega$ is an elliptic curve. In this case, the map ϕ is called a *Lattès map*, a subfamily of which was studied first by Lattès [**12**]. The degree of the map ϕ is $|\lambda|^2$. By the Riemann–Hurwitz formula, the map ψ is unramified. As ramification behaves multiplicatively in compositions, we have $e_\psi(z)\, e_\pi(\psi(z)) = e_\pi(z)\, e_\phi(\pi(z))$, where $e_f(z)$ denotes the ramification index of f at z. So the ramification values of $\phi\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ are a subset of the ramification values of $\pi\colon \mathbf{C}/\Omega \longrightarrow \mathbb{P}^1$. The following theorem is due to Milnor.

THEOREM 2.1. [**13**, Theorem 3.1] *A rational map $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ is a Lattès map if and only if there is a lattice Ω, an affine map $\psi\colon \mathbf{C}/\Omega \to \mathbf{C}/\Omega$ and some subgroup G of* $\mathrm{Aut}(C/\Omega)$ *so that ϕ is conjugate, by a Möbius transformation, to $\psi/G\colon (\mathbf{C}/\Omega)/G \to (\mathbf{C}/\Omega)/G$ sending the equivalence class of a point z under the action of G to the class of $\psi(z)$.*

Let us stress at this point that there are Lattès maps where the map π is not a quotient map of the form $\mathbf{C}/\Omega \to \mathbb{P}^1 = (\mathbf{C}/\Omega)/G$. In particular, π may be of arbitrary high degree. The above theorem implies that in all such cases one can find a full lattice, say $\Omega' \subseteq \mathbf{C}$, so that ϕ is a Lattès map of the form:

$$\begin{array}{ccc} \mathbf{C}/\Omega' & \xrightarrow{\ \psi\ } & \mathbf{C}/\Omega' \\ \Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi'} \\ \mathbb{P}^1 = (\mathbf{C}/\Omega')/G & \xrightarrow{\ \phi\ } & \mathbb{P}^1 = (\mathbf{C}/\Omega')/G \end{array}$$

Table 1 summarizes the automorphism group and possibilities for the map $\pi\colon \mathbf{C}/\Omega \to (\mathbf{C}/\Omega)/G$.

3. Ramification of Lattès maps

Theorem 2.1 and Table 1 allow us to treat each case individually. For future reference, let us note some standard facts concerning elliptic functions.

- The functions $\wp$ and $\wp'$ are related by the differential equation

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3,$$

TABLE 1. Quotient maps from an elliptic curve giving rise to Lattès maps.

Ω	$j(\mathbf{C}/\Omega)$	$\mathrm{Aut}(\mathbf{C}/\Omega)$	G	π
$\mathbf{Z}[\zeta_3]$	0	$\mathbf{Z}/6\mathbf{Z}$	$\mathbf{Z}/6\mathbf{Z}$	$\wp^3$
$\mathbf{Z}[\zeta_3]$	0	$\mathbf{Z}/6\mathbf{Z}$	$\mathbf{Z}/3\mathbf{Z}$	$\wp'$
$\mathbf{Z}[\zeta_3]$	0	$\mathbf{Z}/6\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\wp$
$\mathbf{Z}[\zeta_4]$	1728	$\mathbf{Z}/4\mathbf{Z}$	$\mathbf{Z}/4\mathbf{Z}$	$\wp^2$
$\mathbf{Z}[\zeta_4]$	1728	$\mathbf{Z}/4\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\wp$
Otherwise	$\neq 0$ or 1728	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}/2\mathbf{Z}$	$\wp$

where $g_2 = 60 \sum_{\omega \in \Omega \setminus \{0\}} \omega^{-4}$ and $g_3 = 140 \sum_{\omega \in \Omega \setminus \{0\}} \omega^{-6}$ are the modular invariants of the corresponding elliptic curve.

- The equation $\wp(z) = \wp(w)$ holds if and only if either $z + w \equiv 0 \pmod{\Omega}$ or $z - w \equiv 0 \pmod{\Omega}$.
- The function $\wp$ has two zeros, denoted d_1 and d_2, in $\mathbf{C}/\Omega$ (see [**2, 4**] for further information on their explicit computation).
- The roots of $\wp'$ are $\omega_1/2 = 1/2$, $\omega_2/2 = \tau/2$ and $\omega_3/2 = (1+\tau)/2$, i.e. half-periods. We denote the set of half-periods and zero by $\frac{1}{2}\Omega$.
- The addition formula of $\wp$ [**8**, p. 160] reads

$$\wp(z+w) = -\wp(z) - \wp(w) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)}\right)^2.$$

If we take the limit as w moves towards z we obtain the duplication formula

$$\wp(2z) = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2.$$

Recall that a map $\phi\colon X \to \mathbb{P}^1$, where X is an algebraic curve, is called a Belyi map if it is ramified at most over three points. As a result of Belyi's celebrated theorem, such an X admits a model over $\bar{\mathbf{Q}}$. Our aim is to prove the following theorem in the case of Lattès maps.

THEOREM 3.1. *Let $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a Lattès map induced by $\psi\colon \mathbf{C}/\Omega \to \mathbf{C}/\Omega$ defined as $\psi(z) = \lambda z$. For any λ in the corresponding endomorphism ring, if $\pi = \wp$, then ϕ is a Belyi map if and only if* $\deg(\psi) = 2$ *or* $\deg(\psi) = 4$ *and λ is integral. In all the remaining cases, i.e. when $\pi = \wp'$ or $\wp^2$ or $\wp^3$, ϕ is a Belyi map (see Table 1).*

Let us begin by observing the following lemma.

LEMMA 3.2. *If the degree of ψ is even (respectively odd) then the cardinality of the intersection $\psi^{-1}(z_o) \cap \frac{1}{2}\Omega$ is even (respectively odd) when z_o is either a period or a half-period.*

Proof. Fix some $z_o \in \frac{1}{2}\Omega$. Let $z \in \psi^{-1}(z_o)$ and $w \in \mathbf{C}$ with $z + w \equiv 0 \pmod{\Omega}$. Then, $w \in \psi^{-1}(z_o)$ because $\lambda(z+w) \equiv \lambda z + \lambda w \equiv \psi(z) + \psi(w) \pmod{\Omega}$, and $-z_o \equiv z_o \pmod{\Omega}$. This means that the parity of the set $\psi^{-1}(z_o)$ is determined by the number $|\psi^{-1}(z_o) \cap \frac{1}{2}\Omega|$. We deduce the result using the fact that ψ is unramified. □

3.1. $G \cong \mathbf{Z}/2\mathbf{Z}$

We have $\pi = \wp$ by Theorem 2.1. So the set of ramification values of ϕ is a subset of $R = \{\wp(1/2), \wp(\tau/2), \wp(\omega_3/2), \wp(0) = \infty\}$. Remark that as λ is a quadratic algebraic integer, $|\lambda|^2 \in \mathbf{Z}$. Let us first note the following lemma.

LEMMA 3.3. *Let $z_o \in \frac{1}{2}\Omega$ be an arbitrary element. Then $\psi^{-1}(z_o)$ contains exactly one element of $\frac{1}{2}\Omega$ whenever $|\lambda|^2$ is odd.*

TABLE 2. Ramification of Lattès maps corresponding to the group $G \cong \mathbf{Z}/2\mathbf{Z}$.

$\deg(\psi)$	$\lvert\phi^{-1}(\wp(0))\rvert$	$\lvert\phi^{-1}(\wp(\omega_i/2))\rvert$
$4(\lambda = \pm 2)$	4	2
Even and > 4	$\lvert\lambda\rvert^2/2 + 2$	$\lvert\lambda\rvert^2/2$
Odd	$(\lvert\lambda\rvert^2 - 1)/2 + 1$	$(\lvert\lambda\rvert^2 - 1)/2 + 1$

Proof. We know that $|\psi^{-1}(z_o)|$ is odd; therefore it has to contain an odd number of elements from the set $\frac{1}{2}\Omega$, by Lemma 3.2. Let z, $w \in \psi^{-1}(z_o) \cap \frac{1}{2}\Omega$ be two elements. Then $z = |\lambda|^2 z = \bar{\lambda}\lambda z = \bar{\lambda}\lambda w = |\lambda|^2 w = w$ in $\mathbf{C}/\Omega$. □

The following result gives a complete picture of the ramification data (summarized in Table 2) in this case.

PROPOSITION 3.4. *If* $\deg(\psi) = 2$ *then the corresponding map is equivalent to the map sending* z *to* z^2. *If* $\lambda = \pm 2$ *then* ϕ *is ramified exactly over* $\{\wp(\omega_i/2)\}$ *with the inverse image of each point containing two elements. For non-integral* λ *with* $|\lambda|^2 = 4$, *the map* ϕ *is not a Belyi map. If* $\deg(\psi) = |\lambda|^2 > 4$ *is an even integer, then* ϕ *is ramified exactly over* R *with the numbers* $|\phi^{-1}(\wp(\omega_i/2))|$ *being* $|\lambda|^2/2$ *and* $|\phi^{-1}(\infty)|$ *being* $|\lambda|^2/2 - 2$. *Finally, if* $|\lambda|^2$ *is odd, then* ϕ *is ramified exactly over* R *so that for any* $w \in R$, $|\phi^{-1}(w)|$ *is* $(|\lambda|^2 - 1)/2 + 1$.

Proof. We have $\lambda = \pm 1 \pm \sqrt{-1}$ or $\lambda = (\pm 1 \pm \sqrt{-7})/2$ for the case $|\lambda|^2 = \pm 2$. In these cases, as the only equivalence class of Belyi maps of degree two is $z \mapsto z^2$, all maps are equivalent to this map. More precisely, in the former case $|\phi^{-1}(0)| = |\phi^{-1}(\omega_3/2)| = 2$ and $|\phi^{-1}(\omega_1/2)| = |\phi^{-1}(\omega_2/2)| = 1$ and in the latter case $|\phi^{-1}(0)| = |\phi^{-1}(\omega_2/2)| = 2$ and $|\phi^{-1}(\omega_1/2)| = |\phi^{-1}(\omega_3/2)| = 1$.

Say $|\lambda|^2$ is an odd integer. Then, for any $z_o \in \frac{1}{2}\Omega$ there is exactly one element of $\frac{1}{2}\Omega$, say w_o, in $\psi^{-1}(z_o)$ (Lemma 3.3). For any element $z \in \psi^{-1}(z_o) \setminus \{w_o\}$ there is some $w \in \psi^{-1}(z_o) \setminus \{w_o\}$ so that $z + w = 0$ in $\mathbf{C}/\Omega$. For these two elements we have $\wp(z) = \wp(w)$. This means that $|\phi^{-1}(\wp(z_o))| = (|\lambda|^2 - 1)/2$.

Suppose now that $|\lambda|^2 = 4$. In the non-integral cases, that is for $\lambda = (\pm 1 \pm \sqrt{-15})/2$ and $\lambda = (\pm 3 \pm \sqrt{-7})/2$, one may compute that $|\wp(\psi^{-1}(0))| = 3$. For the half-periods, one has $|\wp(\psi^{-1}(1/2))| = 3$, $|\wp(\psi^{-1}(\tau/2))| = 3$ and $|\wp(\psi^{-1}(\omega_3/2))| = 2$. This implies that the corresponding map ϕ is ramified over four points, and therefore is not a Belyi map. When $\lambda = \pm 2$, we have $\psi^{-1}(0) = \frac{1}{2}\Omega$ and their images under $\wp$ are distinct, meaning 0 is not a ramification point. However, for any $\omega_i/2 \in \mathbf{C}/\Omega$, elements of $\psi^{-1}(\omega_i/2)$ can be paired in such a way that they add up to $0 \in \mathbf{C}/\Omega$, and therefore $|\wp(\psi^{-1}(\omega_i/2))| = 2$. So $|\phi^{-1}(\wp(\omega_i/2))|$ is 2.

More generally, if $\deg(\psi) > 4$ and even, then a similar diagram chasing argument leads to the fact that $|\phi^{-1}(\infty)| = |\lambda|^2/2 + 4$, because in this case $\frac{1}{2}\Omega \subset \psi^{-1}(0)$. □

3.2. $G \cong \mathbf{Z}/3\mathbf{Z}$

In order to admit an order-three automorphism, Ω must be homothetic to the Eisenstein lattice, $\mathbf{Z}[\zeta_3]$. In this case, the map π is $\wp'$ and we have the identity $\wp''(z) = 6(\wp(z))^2$. For

TABLE 3. Ramification of Lattès maps corresponding to the group $G \cong \mathbf{Z}/3\mathbf{Z}$.

	$\lvert\phi^{-1}(\infty)\rvert$	$\lvert\phi^{-1}(\wp'(d_1))\rvert$	$\lvert\phi^{-1}(\wp'(d_2))\rvert$
$\lvert\psi^{-1}(0)\cap\{d_1, d_2\}\rvert = 2$	$(\lvert\lambda\rvert^2-3)/3+3$	$\lvert\lambda\rvert^2/3$	$\lvert\lambda\rvert^2/3$
$\lvert\psi^{-1}(0)\cap\{d_1, d_2\}\rvert = 1$		Impossible	
$\lvert\psi^{-1}(0)\cap\{d_1, d_2\}\rvert = 0$	$(\lvert\lambda\rvert^2-1)/3+1$	$(\lvert\lambda\rvert^2-1)/3+1$	$(\lvert\lambda\rvert^2-1)/3+1$

$i = 1, 2$ the equation $(\wp'(z))^2 = 4(\wp(z))^3 - g_3$ tells us that the value of $\wp'$ at d_i is $\pm\sqrt{-g_3}$. Remark that $\wp'(d_1) \neq \wp'(d_2)$. Indeed, otherwise the degree-three doubly periodic function $F(z) = \wp'(z) - \wp'(d_1)$ would have two triple zeros, d_1 and d_2. We conclude that ϕ is ramified at most over the set $\{\wp'(d_1), \wp'(d_2), \wp'(0)\}$. The following proposition gives the complete ramification behaviour in this case.

PROPOSITION 3.5. *Let $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a Lattès map of degree three. Then ϕ is ramified exactly over the set $\{\wp'(d_1), \wp'(d_2), \wp'(0)\}$. We refer to Table 3 for the precise ramification data.*

Proof. For brevity, let us only show that $|\psi^{-1}(0) \cap \{d_1, d_2\}| = 1$ is not possible. The proofs of the remaining cases have been exemplified in the proof of Proposition 3.4. Indeed, as a result of the fact that the sum of zeros (counting multiplicities) of an elliptic function should be equal to the sum of poles (counting multiplicities), we obtain $d_1 + d_2 \equiv 0 \pmod{\Omega}$. Say $\psi(d_1) = 0 \pmod{\Omega}$. Then

$$\psi(d_2) \equiv \psi(-d_1) \equiv -\psi(d_1) \equiv 0 \pmod{\Omega}. \qquad \square$$

3.3. $G \cong \mathbf{Z}/4\mathbf{Z}$

Here Ω is homothetic to the ring of Gaussian integers, $\mathbf{Z}[\sqrt{-1}]$, and the map π is $\wp^2$. As a result of the extra symmetry of $\wp$, namely $\wp(\sqrt{-1}\, z) = -\wp(z)$, we have $d_1 = d_2 = \omega_3/2$ and $\wp(1/2) = -\wp(\sqrt{-1}/2)$. To determine the ramification points of ϕ we determine the zeros and poles of $\wp$ and $\wp'$. So, the set $R = \{\wp^2(0) = \infty, \wp^2(1/2) = \wp^2(\sqrt{-1}/2), \wp^2(\omega_3/2) = \wp^2(d_1) = \wp^2(d_2) = 0\}$ is the set of all possible ramification values of ϕ. We further have that $\psi^{-1}(0)$ is the lattice generated by $(b + a\sqrt{-1})/(a^2 + b^2)$ and $(a - b\sqrt{-1})/(a^2 + b^2)$, where $\lambda = a + b\sqrt{-1}$. The set $\psi^{-1}(\{0, 1/2, \sqrt{-1}/2, (1 + \sqrt{-1})/2\})$ is also a lattice, denoted by F_λ, generated by halves of the generators of $\psi^{-1}(0)$. That is, $\psi^{-1}(0) = 2F_\lambda$. To ease notation, elements of $\psi^{-1}(0)$ will be referred to as vertices of type $\times$, elements of $\psi^{-1}(1/2) \cup \psi^{-1}(\sqrt{-1}/2)$ (or edge centres) will be referred to as vertices of type $\circ$ and elements of $\psi^{-1}((1 + \sqrt{-1})/2)$ (or face centres) will be referred to as vertices of type $\bullet$. We refer to Figure 1 for an instance ($\lambda = 3 + 4\sqrt{-1}$).

The degree-four map $\wp^2\colon \mathbf{C}/\mathbf{Z}[\sqrt{-1}] \to \mathbb{P}^1$ can be described geometrically: $\wp^2$ identifies the line segments $[0, 1/2]$ and $[1/2, (1 + \sqrt{-1})/2]$ with $[0, \sqrt{-1}/2]$ and $[\sqrt{-1}/2, (1 + \sqrt{-1})/2]$, respectively. To determine the exact ramification we will determine points of the lattice F_λ that lie in $[0, 1/2] \times [0, \sqrt{-1}/2]$, which is denoted by $F_{\lambda,\wp^2}$, paying attention to their types. The points of the set $((0, 1/2) \times (0, \sqrt{-1}/2)) \cap F_{\lambda,\wp^2}$ will be called

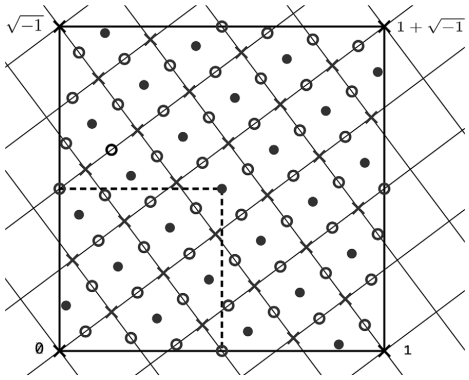


FIGURE 1. Inverse image of the set $\psi^{-1}(0, \infty, \wp^2(1/2))$ in F_λ for $\lambda = 3 + 4\sqrt{-1}$: $\times$, $\circ$ and $\bullet$ represent elements of $\psi^{-1}(0)$, $\psi^{-1}(1/2) = \psi^{-1}(\sqrt{-1}/2)$ and $\psi^{-1}((1+\sqrt{-1})/2)$, respectively.

interior lattice points, and points that are in $\partial([0, 1/2] \times [0, \sqrt{-1}/2]) \cap F_{\lambda,\wp^2}$ are called boundary lattice points.

As $\mathbf{Z}[\sqrt{-1}]$ has multiplication by $\sqrt{-1}$ symmetry, without loss of generality we will assume that $\lambda = a + b\sqrt{-1}$, with $a, b \geq 0$. We would like to remark that elements of F_λ in the fundamental region of $F = \{z \in \mathbf{C} \mid 0 \leq \mathrm{Re}(z), \mathrm{Im}(z) \leq 1\}$ is exactly the set of intersection points of the lines

$$2by = 2ax + j, \quad j = -2a, 1-2a, \ldots, 2b-1, 2b,$$

and

$$2ay = -2bx + k, \quad k = 0, 1, \ldots, 2(a+b).$$

Let us collect some preliminary results.

(PR1) The types of vertices of $F_{\lambda,\wp^2}$ depend on the parities of the integers a and b. Table 4 gives a summary where each claim can be seen by direct computation.

(PR2) Each one of the four line segments bounding the square $[0, 1/2] \times [0, \sqrt{-1}/2]$ can contain at most two distinct vertex types. Indeed, the parities of the integral solution of the equation $a'\alpha + b'\beta = 0$ (where $a = \gcd(a, b)a'$ and $b = \gcd(a, b)b'$) determine the type of the corresponding element $F_{\lambda,\wp^2}$ on the segment $[0, 1/2]$. Precisely, the vertex is of type $\times$ (respectively $\bullet$) if α and β are both even (respectively odd). A vertex of type $\circ$ is on the segment $[0, 1/2]$ if α and β have opposite parities.

TABLE 4. Parities of a and b and values of ψ at half-periods.

a	b	$1/2 \in$	$\sqrt{-1}/2 \in$	$(1+\sqrt{-1})/2 \in$
Even	Even	$\times$	$\times$	$\times$
Even	Odd	$\circ$	$\circ$	$\bullet$
Odd	Even	$\circ$	$\circ$	$\bullet$
Odd	Odd	$\bullet$	$\bullet$	$\times$

(PR3) If z is an element of $F_{\lambda,\wp^2}$, then $z+1/2$ and $z+\sqrt{-1}/2$ are also a vertex of $F_{\lambda,\wp^2}$. If a and b are both even then the type of the vertices $z+1/2$ and $z+\sqrt{-1}/2$ is the same as that of z. A vertex of type $\times$ becomes a vertex of type $\circ$ if a and b are of opposite parities and a vertex of type $\bullet$ if both a and b are odd. A vertex of type $\bullet$ becomes a vertex of type $\circ$ if a and b are of opposite parities and a vertex of type $\times$ if both a and b are odd. A vertex of type $\circ$ becomes a vertex of type $\times$ if a and b are of opposite parities. We cannot have a vertex of type $\circ$ along the boundary in the other cases. To exemplify the argument, let us suppose that a is even and b is odd with $z=p/q$ (with $p, q \in \mathbf{Z}$ and relatively prime) so that $\psi(z)=1/2$. Then we must have $(a+b\sqrt{-1})p/q = 1/2$ implying $ap/q \equiv 1/2 \pmod 1$. This means $4 \mid q$, as a is even. Therefore q is even. On the other hand, the vanishing of the imaginary part implies $q \mid b$, which is a contradiction as b is odd. The other cases are treated similarly.

(PR4) The family of lines $2by = 2ax + j$, $j = 0, \ldots, 2b-1$, $2b$, divide $[0, 1]$ into $2b$ equal intervals, and the lines $2ay = -2bx + k$, $k = 0, 1, \ldots, 2a$, into $2a$ equal intervals. In particular, an element of $F_{\lambda,\wp^2}$ is a point on $[0, 1]$ whenever $0 \le \gamma/(2a) = \delta/(2b) \le 1$ for some non-negative integers γ and δ.

LEMMA 3.6. *For $\lambda = a + b\sqrt{-1}$, we set $d = \gcd(a, b)$ and $a = da'$ and $b = db'$. The number of vertices of types $\times$, $\circ$ and $\bullet$ in $F_{\lambda,\wp^2}$ depend on the parities of a, b, a' and b'. See Table 5 for the results.*

TABLE 5. Parities of a and b and types of vertices in the interior and on the boundary.

Parities of a and b	Vertex in	Type $\times$ vertices	Type $\circ$ vertices	Type $\bullet$ vertices
Both even and a', b' odd	int	$\lvert\lambda\rvert^2/4 - d + 1$	$\lvert\lambda\rvert^2/2$	$\lvert\lambda\rvert^2/4 - d$
	∂	$2d$	0	$2d$
Both even and a', b' mixed	int	$\lvert\lambda\rvert^2/4 - d + 1$	$\lvert\lambda\rvert^2/2 - d$	$\lvert\lambda\rvert^2/4$
	∂	$2d$	$2d$	0
Mixed	int	$(\lvert\lambda\rvert^2 - 2d + 1)/4$	$(\lvert\lambda\rvert^2 - 2d + 1)/2$	$(\lvert\lambda\rvert^2 - 2d + 1)/4$
	∂	d	$2d$	d
Both odd	int	$(\lvert\lambda\rvert^2 + 2)/4 - d$	$\lvert\lambda\rvert^2/2$	$(\lvert\lambda\rvert^2 + 2)/4 - d$
	∂	$2d$	0	$2d$

Proof. We will first assume that both a and b are even. To determine elements of $F_{\lambda,\wp^2}$ we will use Pick's theorem, which predicts that, given such a lattice polygon (i.e., a polygon whose vertices are elements of the lattice), one has $A = i + j/2 - 1$, where A is the number of lattice squares in the polygon, i is the number of interior points and j is the number of boundary points. Among these terms, remark first that there are $|\lambda|^2/4$ lattice squares because $\deg(\psi) = |\lambda|^2$. Then (PR1) implies that the vertices of $F_{\lambda,\wp^2}$ are exactly at 0, $1/2$, $\sqrt{-1}/2$, $(1+\sqrt{-1})/2$. (PR3) implies that it is enough to count the vertices along the segment $[0, 1/2]$ to determine j. As discussed in (PR2) such lattice points correspond to integral solutions of the equation $a'\alpha + b'\beta = 0$, for which the quantity $(b'\alpha - a'\beta)/[2d((a')^2 + (b')^2)] \in [0, 1]$. The method described in (PR4), which counts the number of points on $[0, 1]$, implies that the number of such solutions is exactly $2d + 1$. From here, we conclude that $j = 4d$. We

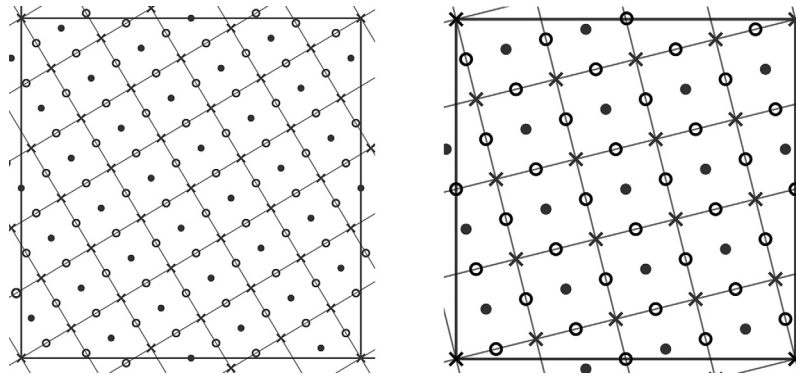


FIGURE 2. The set $F_{\lambda,\wp^2}$ when $\lambda = 6 + 10\sqrt{-1}$ and $\lambda = 2 + 8\sqrt{-1}$, respectively.

deduce $i = |\lambda|^2/4 - 4d/2 + 1$. As both a and b are even, $z \mapsto \frac{1}{2}\lambda z$ also defines a Lattès map for which we have $F_{\frac{1}{2}\lambda} \cap F = F_{\lambda,\wp^2}$; therefore there are exactly $|\lambda|^2/4$ many face centres (vertices of type $\bullet$) in $F_{\lambda,\wp^2}$. Each edge of any square is incident to exactly two squares; therefore, in total there are $|\lambda|^2/2$ many edge centres in $F_{\lambda,\wp^2}$. Among the face centres, only when a' and b' are both odd, do $2d$ of them lie on the boundary of $F_{\lambda,\wp^2}$. Otherwise, that is when a' and b' are of mixed parity, there are $2d$ many vertices of type $\circ$ on the boundary; see Figure 2. We obtain the number of vertices of type $\times$ on the boundary as $2d$. So, we must have $|\lambda|^2/4 - d + 1$ many interior lattice points of type $\times$.

In the case where both a and b are odd we first note that the boundary lattice points can only be of type $\times$ and $\bullet$ and they always exist, but we cannot have any vertex of type $\circ$ on the boundary. We pass to 2λ, and apply the results obtained above to obtain the number of interior lattice points of $F_{2\lambda,\wp^2}$ of type $\times$ (respectively $\circ$ and $\bullet$) as $|\lambda|^2 - 2d + 1$ (respectively $2|\lambda|^2$ and $|\lambda|^2 - 2d$). The interior lattice points of $F_{2\lambda,\wp^2}$ contain four copies of interior lattice points of $F_{\lambda,\wp^2}$ and four copies of boundary lattice points. Therefore, the number of interior lattice points of $F_{\lambda,\wp^2}$ must be $(|\lambda|^2 - 2d)/4 - (d-1)/2 = (|\lambda|^2 + 2)/4 - d$. In a similar fashion, the number of points of type $\bullet$ (respectively $\circ$) are counted as $(|\lambda|^2 + 2)/4 - d$ (respectively $|\lambda|^2/2$).

For the final case where the parities of a and b are opposite, one computes the points in $F_{\lambda,\wp^2}$ by comparing it with $F_{2\lambda,\wp^2}$. We leave the proof to the reader, as no new arguments are involved. □

The following result gives the precise ramification of Lattès maps in question.

PROPOSITION 3.7. *Let $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a Lattès map with the degree of π being four. Then we have the results in Table 6.*

Proof. Let us treat the case where both a and b are odd and leave the remainder of the proof to the reader. We begin by noting that

$$|\phi^{-1}(\infty)| = |\{\wp^2(z) \mid z \text{ is a vertex of type } \times\}|.$$

TABLE 6. Ramification of Lattès maps corresponding to the group $G \cong \mathbf{Z}/4\mathbf{Z}$.

Parities of a and b	$\lvert\phi^{-1}(\infty)\rvert$	$\lvert\phi^{-1}(\wp^2(1/2))\rvert$	$\lvert\phi^{-1}(0)\rvert$
Both even	$\lvert\lambda\rvert^2/4+2$	$\lvert\lambda\rvert^2/2$	$\lvert\lambda\rvert^2/4$
Mixed	$(\lvert\lambda\rvert^2+3)/4$	$(\lvert\lambda\rvert^2+1)/2$	$(\lvert\lambda\rvert^2+3)/4$
Both odd	$(\lvert\lambda\rvert^2+2)/4+1$	$\lvert\lambda\rvert^2/2$	$(\lvert\lambda\rvert^2+2)/4$

By Lemma 3.6, there are $(a^2+b^2+2)/4-d$ many distinct images of vertices of type $\times$ which are in the interior. For the boundary lattice points, as a result of the proof of Lemma 3.6, the $2d$ many points give rise to $2(d-1)/2+2=d+1$ distinct values under $\wp^2$. Again, using the fact that $|\phi^{-1}(0)| = |\{\wp^2(z) \mid z \text{ is a vertex of type } \bullet\}|$, we find the number of the images of inner lattice points as $(|\lambda|^2+2)/4-d$. Concerning the boundary lattice points of type $\bullet$, we obtain $2(d-1)/2+1=d$. □

The norm-one case, that is $\lambda=\pm1$ or $\lambda=\pm\sqrt{-1}$, does not lead to any interesting Lattès map. Also $\lambda=\pm1\pm\sqrt{-1}$ is not interesting for our purposes as, in these cases, the map ϕ is of degree two and by Proposition 3.7 ϕ is ramified exactly over 0 and $\wp^2(1/2)$. In all the remaining cases, ϕ has exactly three ramification values.

3.4. $G \cong \mathbf{Z}/6\mathbf{Z}$

This is possible only when the lattice is the Eisenstein integers, $\mathbf{Z}[\zeta_3]$. The map π becomes $\wp^3$. Candidates for ramification points are then $0, d_1=\omega_3/3, d_2=2\omega_3/3, \omega_1/2, \omega_2/2, \omega_3/2$. We also have $\wp^3(\omega_i/2)=g_3/4$, as $g_2=0$. Therefore, there are only three candidates for ramification values, namely $\wp^3(d_i)=0$, $\wp^3(\omega_i/2)=g_3/4$ and $\wp^3(0)=\infty$.

As the map $\wp^3$ is invariant under the action of $\mathbf{Z}/6\mathbf{Z}$, which acts as multiplication by ζ_6 on $\mathbf{C}/\mathbf{Z}[\zeta_3]$, the set $F_{\lambda,\wp^3}$ becomes the finite closed region whose boundary contains the lines $\arg(z)=0$, $\arg(z)=\pi/3$, $\mathrm{Re}(z)=1/2$ and $\arg(z-1)=2\pi/3$; see Figure 3.

One may follow a line of reasoning similar to the $\mathbf{Z}/4\mathbf{Z}$ case. More precisely, vertices of the fundamental region of $\wp^3$ are determined by values of a and b (mod 6). (PR2) does not have an analogue in this case; see the $\lambda=1+4\zeta_6$ case in Figure 4. (PR3) and (PR4) have analogues with small modifications. All these results lead to a Lemma 3.6 type statement. Consequently, we obtain the following proposition.

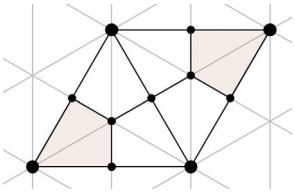


FIGURE 3. Shaded region is the fundamental region of $\wp^3$.

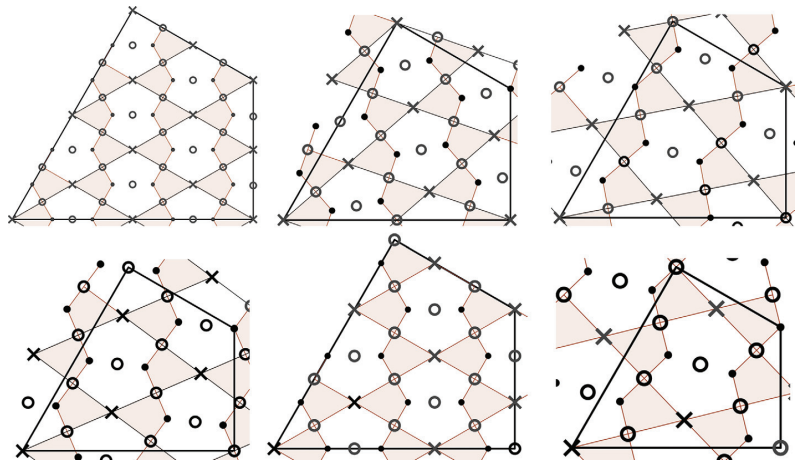


FIGURE 4. The set $F_{\lambda,\wp^3}$ when $\lambda = 4+4\zeta_6, 4+2\zeta_6, 1+4\zeta_6, 2+3\zeta_6, 3+3\zeta_6, 1+3\zeta_6$, respectively.

PROPOSITION 3.8. *Let $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a Lattès map with the degree of π being six and let $\lambda = a + b\zeta_6$. Then the exact ramification of ϕ depends on both the parities of a and b as well as on the value of $b-a$ in $\mathbf{Z}/6\mathbf{Z}$; see Table 7.*

TABLE 7. Ramification of Lattès maps corresponding to the group $G \cong \mathbf{Z}/6\mathbf{Z}$.

Parities of a and b	$\lvert\phi^{-1}(\infty)\rvert$	$\lvert\phi^{-1}(\wp^3(\omega_i/2))\rvert$	$\lvert\phi^{-1}(0)\rvert$
Both even and $a \equiv b \pmod 6$	$\lvert\lambda\rvert^2/6+2$	$\lvert\lambda\rvert^2/2$	$\lvert\lambda\rvert^2/3$
Both even and $a \not\equiv b \pmod 6$	$(\lvert\lambda\rvert^2+8)/6$	$\lvert\lambda\rvert^2/2$	$(\lvert\lambda\rvert^2+2)/3$
Mixed and $b-a \equiv 3 \pmod 6$	$(\lvert\lambda\rvert^2+9)/6$	$(\lvert\lambda\rvert^2+1)/2$	$\lvert\lambda\rvert^2/3$
Mixed and $b-a \not\equiv 3 \pmod 6$	$(\lvert\lambda\rvert^2+5)/6$	$(\lvert\lambda\rvert^2+1)/2$	$(\lvert\lambda\rvert^2+2)/3$
Both odd and $a \equiv b \pmod 6$	$(\lvert\lambda\rvert^2+9)/6$	$(\lvert\lambda\rvert^2+1)/2$	$\lvert\lambda\rvert^2/3$
Both odd and $a \not\equiv b \pmod 6$	$(\lvert\lambda\rvert^2+5)/6$	$(\lvert\lambda\rvert^2+1)/2$	$(\lvert\lambda\rvert^2+2)/3$

The proof follows a similar reasoning as the proof of Proposition 3.7 and therefore will be omitted. Namely, one notes that the inverse image of the lattice $\mathbf{Z}[\zeta_3]$ is again a lattice and relies on Pick's theorem to compute the lattice points in the fundamental region of $\wp^3$. We conclude that the map ϕ has exactly three ramification values apart from the norm-one case.

4. Examples

4.1. $G = \mathbf{Z}/3\mathbf{Z}$

We consider ψ to be the multiplication by $n \in \mathbf{Z}_+$ map on $\mathbf{C}/\Omega$, where $\Omega = \mathbf{Z}[\zeta_3]$. The corresponding Lattès map, denoted $\phi_{3,n}$, is ramified at most over three points, and hence a Belyi map. These Belyi Lattès maps are visualized as gluing two copies of an equilateral triangle whose edges are divided into n equal segments; see Figure 5.

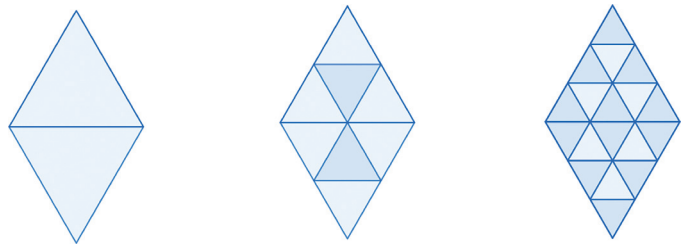


FIGURE 5. First three subdivisions of an equilateral triangle.

The corresponding Belyi maps can be computed, using duplication formulae, as:

$$\phi_{3,2}(z) = \frac{(z-1)(z+1)^3}{(2z-1)^3},$$

$$\phi_{3,3}(z) = \frac{1}{27}\frac{(z^3+3z^2-6z+1)^3}{z(z-1)(z^2-z+1)^3},$$

$$\phi_{3,4}(z) = \frac{(z-2)^3 z(z^4+10z^3-12z^2+4z-2)^3}{(2z-1)^3(2z^4-4z^3+12z^2-10z-1)^3},$$

$$\phi_{3,5}(z) = \frac{(z-1)(z^8+17z^7-107z^6+164z^5-155z^4+164z^3-107z^2+17z+1)^3}{(5z^8-20z^7+125z^6-305z^5+275z^4-65z^3-40z^2+25z-1)^3},$$

$$\phi_{3,6}(z) = \frac{1}{27}\frac{(z^3-6z^2+3z+1)^3(z^9+36z^8-99z^7+165z^6-387z^5+666z^4-564z^3+225z^2-45z+1)^3}{(z-2)^3(z-1)z(z+1)^3(2z-1)^3(z^2-z+1)^3(z^6-3z^5+60z^4-115z^3+60z^2-3z+1)^3}.$$

We remark that each irreducible factor of the rational function $\phi_{3,n}$ is defined over a proper subfield of the $2n$-division field of the corresponding elliptic curve, which is in general not a Galois extension of $\mathbf{Q}$.

For non-integer λ one obtains Belyi maps, as well. For instance, if we take $\lambda = 2+\sqrt{-3}$ we obtain

$$\phi_{3,\lambda}(z) = (\bar{\lambda})^3 \frac{(z-1)(z^2+(-1+3\zeta_3)z+1)^3}{(7z^2-7z-(2+3\zeta_3))^3}.$$

4.2. $G = \mathbf{Z}/4\mathbf{Z}$

Let us again consider the multiplication by n map on $\mathbf{C}/\Omega$, where $\Omega = \mathbf{Z}[\sqrt{-1}]$. The corresponding Belyi maps can be obtained by gluing two copies of isosceles right triangles along their boundary which is tiled using squares; see Figure 6.

Duplication formulae of the corresponding $\wp$ help one to compute these Belyi Lattès maps explicitly, as exemplified below:

$$\phi_{4,2}(z) = \frac{1}{16}\frac{(z+1)^4}{(z-1)^2 z},$$

$$\phi_{4,3}(z) = \frac{z(z^2+6z-3)^4}{(3z^2-6z-1)^4},$$

$$\phi_{4,4}(z) = \frac{1}{256}\frac{(z^4+20z^3-26z^2+20z+1)^4}{(z-1)^2 z(z+1)^4(z^2-6z+1)^4},$$

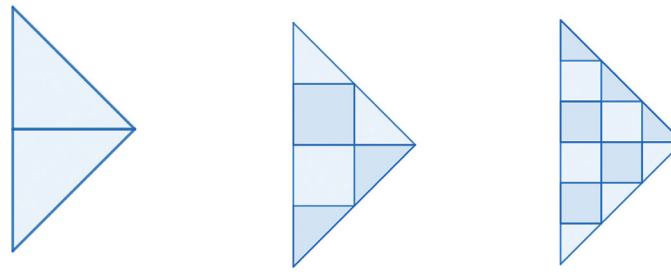


FIGURE 6. First three subdivisions of an isosceles right triangle.

$$\phi_{4,5}(z) = \frac{z(z^2-2z+5)^4(z^4+52z^3-26z^2-12z+1)^4}{(5z^2-2z+1)^4(z^4-12z^3-26z^2+52z+1)^4},$$
$$\phi_{4,6}(z) = \frac{1}{16}\frac{(z+1)^4(z^8+104z^7-548z^6+3032z^5-4922z^4+3032z^3-548z^2+104z+1)^4}{(z-1)^2z(z^2+6z-3)^4(3z^2-6z-1)^4(z^4-28z^3+6z^2-28z+1)^4}.$$

One may extend the above computations to non-integer λ. For instance, when we take $\lambda = 2+\sqrt{-1}$, we obtain

$$\phi_{4,\lambda}(z) = \frac{z(z+(-1+2\sqrt{-1}))^4}{((2+\sqrt{-1})z-\sqrt{-1})^4}.$$

4.3. $G = \mathbf{Z}/6\mathbf{Z}$

Let us again consider the multiplication by n map on $\mathbf{C}/\Omega$, where $\Omega = \mathbf{Z}[\zeta_3]$. The corresponding Belyi maps can be obtained by tiling an equilateral triangle by $\pi/6-\pi/3-\pi/2$ triangles. We exemplify below the corresponding Belyi Lattès maps:

$$\phi_{6,2}(z) = \frac{1}{64}\frac{z(z+8)^3}{(z-1)^3},$$
$$\phi_{6,3}(z) = \frac{1}{729}\frac{(z^3+96z^2+48z-64)^3}{(z-4)^6z^2},$$
$$\phi_{6,4}(z) = \frac{1}{4096}\frac{z(z+8)^3(z^4+536z^3-1344z^2+2048z-512)^3}{(z-1)^3(z^2-20z-8)^6},$$
$$\phi_{6,5}(z) = \frac{z(z^8+2080z^7+50320z^6-367040z^5+2924800z^4-3491840z^3+2805760z^2-1064960z-327680)^3}{(5z^4-380z^3-240z^2+1600z-256)^6}.$$

Let us also exemplify a non-integer entry with $\lambda = 2+\sqrt{-3}$:

$$\phi_{6,\lambda}(z) = (\bar{\lambda})^6\frac{z(z^2-(44+48\zeta_3)z+16+48\zeta_3)^3}{(7z+8+12\zeta_3)^6}.$$

All Belyi maps are computed in PARI [**16**]. One may compute Belyi maps of arbitrarily large degree using the method described. Moreover, there are many interesting relations among the irreducible factors that appear in the numerator and denominator of Belyi Lattès maps. We leave such problems to the interested reader. As mentioned before, one prefers

to study these maps as they are easy to compute; relevant PARI/gp routines are available on the author's webpage (http://math.gsu.edu.tr/azeytin/pdfs/belyi.txt). For instance, the computation of the Belyi Lattès map of degree 29^2 took 5 min, 35,316 ms on a 10 year old laptop of 2.4 GHz Quad-Core Intel Core i7 processor with 8 GB memory. Notice that 29 is chosen in order to make sure that the numerator is an irreducible polynomial of degree 29^2.

5. Galois action on Belyi Lattès maps

Let us end the paper with a discussion of the Galois action on Belyi Lattès maps. The advantage of Belyi Lattès maps over other suggested families (e.g. trees, regular dessins, etc.) lies in its computability, in the sense that one may explicitly write and study rigorously arbitrarily high-degree Belyi Lattès maps, using the general formulae given above.

Notice that for $\lambda = a + b\sqrt{-1} \in \mathbf{C}$ satisfying $\lambda\Omega \subseteq \Omega$, points of type $\bullet$, $\circ$ and $\times$ are all torsion points. For instance, when $\Omega = \mathbf{Z}[\sqrt{-1}]$ points of type $\bullet$ and $\circ$ are of order $2|\lambda|^2/\gcd(a, b)$ and points of type $\times$ are of order $|\lambda|^2/\gcd(a, b)$. To this end, we let E be the elliptic curve whose j invariant is either 0 or 1728 and let λ be an element in the endomorphism ring of E. We take $K = \mathbf{Q}(\sqrt{-1})$ when $j(E) = 0$ and $K = \mathbf{Q}(\sqrt{-3})$ when $j(E) = 1728$. We set n_λ to be the smallest positive integer so that elements of F_λ are a subset of the set of points of order n_λ on E, that is the set $E[n_\lambda]$. The following result is a direct consequence of [**11**, Theorem 2, p. 126].

THEOREM 5.1. *The field of definition of the corresponding Belyi Lattès map is a subfield of the Ray class field of K of conductor n_λ.*

The Belyi Lattès maps arising from the subfamily of multiplication by n maps gives rise to functions whose ramification set lies in a division field of $\wp$. This is in a sense the optimal case as one has $n_\lambda = n$ and $E[n] = F_\lambda$. For n sufficiently large, these fields are Galois extensions of $\mathbf{Q}$. The Galois groups are $\mathrm{GL}(2, \mathbf{Z}/n\mathbf{Z})$ [**15**].

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